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The equivalence of the DLA and a hydrodynamic model

Zbigniew Koza

Institute of Theoretical Physics, University of Wrocław, ul. Cybulskiego 36, PL-50-205 Wrocław, Poland

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Abstract. We prove that the growth process governed by the diffusion-limited aggregation (DLA) algorithm as well as by some of its natural extensions is isomorphous to the process of pushing an incompressible Newtonian fluid by an inviscid fluid through a porous medium in the chamber–tube (CT) model, provided that the parameters of the CT model are properly chosen. In this way we give a direct quantitative relation between hydrodynamics in random (porous) media and the fractal theory of stochastic DLA-like aggregates. On the other hand, this work presents an example of two equivalent dynamical systems, one of them with stochastic dynamics applied to a deterministic medium and the other with deterministic dynamics applied to a random medium.

1. Introduction

In the past few years one can observe a growing effort to understand physical properties of stochastic aggregates, i.e. the structures with stochastic rules of growth. Though this problem elicits interest from many applied sciences and technologies, its theory is still incomplete and in fact even for many well-defined mathematical models it is confined mainly to hypotheses based on numerical simulations. One of the reasons is that this problem belongs to non-equilibrium physics. On the other hand, it has challenged many mathematicians and physicists ever since it was realized that Mandelbrot's concepts of fractal behaviour might be used to solve it. In this paper we rigorously connect two apparently different types of stochastic growth: diffusion-limited aggregation (DLA) and a two-phase flow through a porous medium (the chamber–tube (CT) model).

The DLA (Witten and Sander 1981, 1983) is a well known lattice model with the following recursive rules of growth:

(i) Initially a seed particle is placed at the origin of the lattice, forming a cluster which occupies one lattice site.

(ii) Given a cluster occupying n lattice sites, a cluster consisting of $n+1$ particles is obtained by launching a particle from 'far away' and letting it walk at random until it arrives at any of the sites adjacent to the cluster (the perimeter sites)—then it is stopped and attached to the cluster.

Tang (1985) modified the DLA algorithm by assuming that at each perimeter site j there is a 'counter' c_j indicating how many times this site has been already visited by a random walker. Only if $c_j = M$, M being a previously chosen parameter, is site j incorporated into the aggregate. It is easy to see that this modification is equivalent to letting each lattice site be filled by up to M particles at the same time. Then c_j would correspond simply to the number of particles incorporated in the cluster at j, and the 'kernel' of the cluster would be characterized by the condition $c_j = M$.

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In this paper we will consider also two rather rarely employed extensions of the DLA. In the classical DLA algorithm the probability α_{ij} that a random walker will move from site i to its neighbour j is equal to $1/n_i$, where n_i is the number of the nearest neighbours of i. Following Meakin (1987) we may assume more generally that this probability is proportional to some previously chosen number k_{ij}—see (2.3). Choosing $k_{ij} \equiv 1$ we get $\alpha_{ij} = 1/n_i$. Another extension we are going to consider is the assumption that when a random walker arrives at a perimeter site j, it joins the cluster with probability γ_{jn}, $n \equiv c_j$, and with probability $1 - \gamma_{jn}$ the random walking is started again without any change to the cluster. The classical DLA may be obtained when $\gamma_{jn} \equiv 1$.

Paterson (1984) noticed that the DLA algorithm might be employed in hydrodynamics to predict the behaviour of an inviscid fluid being injected into a porous medium previously completely filled with a viscous fluid. Since then this problem has been explored and discussed by several authors (Lenormand 1989, Måløy *et al* 1987, Frette *et al* 1990, Chen and Wilkinson 1985, Vicsek 1987). For the purpose of this paper the most important conclusion is that the patterns obtained by the DLA algorithm for $M = 1$ resemble those received in the hydrodynamic experiment only in the case of extremely random porous medium, whereas in the limit $M \to \infty$ ('mean-field limit', Witten and Sander 1983) this algorithm corresponds to (Tang 1985) the Saffman–Taylor problem of two incompressible Newtonian fluids, an inviscid one driving a viscous one from a thin space confined by two plates (a Hele–Shaw cell).

Recently, a new approach was developed (Chan *et al* 1988) to model a two-fluid flow in a porous medium. This formalism incorporates quantitatively the random nature of a real porous medium as well as macroscopic equations of hydrodynamics. Namely, it is assumed that a porous medium can be regarded as a system of chambers (pores) connected by tubes. In contrast to many previous models, the chamber volumes ϕ_i (local porosities) and the hydrodynamic conductivities (local permeabilities) of the tubes k_{ij} are random variables (but in this paper we assume that only ϕ_i have random nature). The tubes hold a negligible volume of the fluids, but give rise to a hydrodynamic drag, wheras the chambers hold virtually all of the fluids, but make negligible contribution to the hydrodynamic resistance. A flow of a Newtonian incompressible fluid is governed by Darcy's law (3.4), and by a simple condition of incompressiblitiy, (3.5). We shall consider only the case of an inviscid fluid driving an incompressible viscous one at constant pressure, which provides us with simple boundary conditions (3.3*a*, *b*).

In sections 2 and 3 we give more precise definitions of the two models. We also derive there some useful equations which we will use in section 4 to show the conditions for the equivalence of the models. In section 4 we also present some applications of our theorem: the equivalence of the mean-field limit of the DLA and the Saffman–Taylor problem, and the correspondence between the DLA noise and the CT entropy. We also show how to introduce real-time dependence into the DLA algorithm which would correspond to the CT time variable. Finally, section 5 is devoted to the conclusions.

2. The extended DLA model

Consider a graph $G(Z, Z', \hat{A}, \{k_{ij}\}, \{\gamma_{ij}\}, M)$, where $Z \cup Z'$ is the set of its nodes, $\hat{A}$ is the contiguity matrix ($A_{ij} = 1 \leftrightarrow i, j \in Z \cup Z'$ are connected by a bond and $A_{ij} = 0$ otherwise), $k_{ij} = k_{ji} > 0$ are real weights assigned to the bonds, $M \geqslant 1$ is an integer parameter and for every $i \in Z$ $\gamma_{i0}, \ldots, \gamma_{iM-1}$ is a sequence of positive weights assigned to i. From the physical point of view, Z corresponds to the system, Z' is a special set

from which random walkers are launched into the system, $\{k_{ij}\}$ governs the random walk probabilities, and $\{\gamma_{in}\}$ are the probabilities of attaching to the cluster the $(n+1)$th random-walking particle at $i \in Z$. Usually graph G corresponds to a square, hexagonal or cubic lattice, the system Z consists of all sites lying within a circle (sphere) with an extremely large radius R, and Z' is composed of all sites lying off Z. In this paper, however, Z' is a *one-element* set of nodes, the only element of which is connected to all $i \in Z$, that would be connected with Z' in the usual (infinite lattice) approach. This assumption does not affect the behaviour of the system, but significantly simplifies the considerations. The only element of Z' will be denoted by l or L.

Let $[c(s)]_{s=0}^{\infty}$ be a stochastic Markov process defined by the conditional probability $p(c(s)|c(s-1))$, where s is a 'discrete time' variable and $c(s)$ is a function (field) $c(s)$: $Z \ni i \mapsto c_i(s) \in \{0, 1, \ldots, M\}$. We interpret $c_i(s)$ as the number of particles belonging to the cluster at site i and at time s (the counter variable). Given $c(s)$, we may divide Z into three disjoint subsets: the aggregate $Z^{A}(s)$, the free space $Z^{F}(s)$ and the perimeter $Z^{P}(s)$:

$$Z^{A}(s) \equiv \{i \in Z: c_i(s) = M\} \tag{2.1a}$$

$$Z^{F}(s) \equiv \{i \in Z: \quad c_i(s) = 0 \text{ and } \forall_{j \in Z} \quad A_{ij} = 1 \to c_j(s) = 0\} \tag{2.1b}$$

$$Z^{P}(s) \equiv Z - Z^{A}(s) - Z^{F}(s). \tag{2.1c}$$

Given one of these sets, the other two are fully determined. Let also

$$K_i \equiv \sum_j k_{ij} \tag{2.2}$$

where Σ_j denotes the sum over all j linked with i.

In the DLA model the conditional probability $p(c(s)|c(s-1))$ is defined in the following, descriptive way. Assume that $c(s-1)$ is given. Then we can determine $Z^{A}(s-1)$, $Z^{P}(s-1)$ and $Z^{F}(s-1)$. Let a particle be put at $l \in Z'$ and let it walk at random in accordance with the following rules:

(i) If the particle is at $i \in Z^{F}(s-1) \cup Z'$, then it moves to any of its neighbours $j \in Z \cup Z'$ with probability α_{ij},

$$\alpha_{ij} \equiv k_{ij}/K_i. \tag{2.3}$$

(ii) If the particle arrives at $k \in Z^{P}(s-1)$, it is stopped and attached to the cluster with probability γ_{kn}, $n \equiv c_k(s-1)$, otherwise it is restarted from Z' without any change to the cluster.

From this description it follows immediately that the growth can occur only at exactly one of the perimeter nodes and that mathematically it is expressed by increasing c_i by 1 at exactly one node. Thus, denoting this growth node by k, we have

$$c_i(s) = c_i(s-1) + \delta_{ik} \qquad i \in Z \tag{2.4}$$

as the condition for a non-zero value of $p(c(s)|c(s-1))$. Henceforth we shall denote these non-zero conditional probabilities simply by $p_s(k)$, remembering that they depend not only on k, but on $c(s-1)$ as well. By definition

$$p_s(k) = \gamma_{kn} * \text{prob (walker arrived at } k). \tag{2.5}$$

Now let us express $p_s(k)$ in terms of the parameters of the model. Again let $c(s-1)$ be given and let also $\rho(N, i)$ be the probability of finding the random walker at

$i \in Z \cup Z'$ after having performed N random steps. Let also $\chi_i = 1 \leftrightarrow i \in Z^F(s-1) \cup Z'$ and $\chi_i = 0$ otherwise. Then directly from the definition of the random walk process,

$$\begin{aligned} &\forall_{N \geq 1} \quad \forall_{i \in Z \cup Z'} \qquad \rho(N, i) = \sum_j \mathcal{H}_j \rho(N-1, j) \alpha_{ji} \\ &\qquad\qquad\qquad\qquad \rho(0, i \in Z) = 0 \\ &\qquad\qquad\qquad\qquad \rho(0, l \in Z') = 1. \end{aligned} \tag{2.6}$$

The aim of introducing χ_j is to express the fact that the random walker cannot return to the free space $Z^F(s-1)$ after arriving at the perimeter $Z^P(s-1)$. Let

$$\rho_i \equiv \sum_{N=0}^{\infty} \rho(N, i). \tag{2.7}$$

This quantity is the mean number of times the site i is being visited by a random walker on its way to the cluster, and corresponds to the density of mutually independent random walkers being launched from Z' at a constant rate and annihilated at the perimeter without any influence on the cluster. If k is one of the perimeter nodes, all $\rho(N, k)$ are mutually independent and so

$$\rho_k = \text{prob (walker arrived at } k) \qquad k \in Z^P(s-1). \tag{2.8}$$

In terms of ρ_i, (2.6) reads

$$\rho_i = \sum_j \chi_j \rho_j k_{ji} / K_j \qquad i \in Z \tag{2.9a}$$

$$\rho_l = \sum_j \chi_j \rho_j k_{jl} / K_j + 1 \qquad l \in Z'. \tag{2.9b}$$

To simplify these equations we locally rescale ρ:

$$\rho_i^* \equiv \chi_i (\rho_i / K_i)(K_L / \rho_L) \qquad i \in Z \cup Z', L \in Z'. \tag{2.10}$$

Then (2.9a) yields

$$\rho_i^* = \sum_j \rho_j^* k_{ji} / K_i \qquad i \in Z^F(s-1). \tag{2.11}$$

In the case of a uniform system, $k_{ij} = \text{const}$, (2.11) becomes a discrete-lattice version of the Laplace equation $\Delta \rho^* = \Delta \rho = 0$. In the general case the boundary conditions are provided by (2.10) and the definition of χ_i,

$$\begin{aligned} &\rho_i^* = 0 \qquad i \in Z^A(s-1) \cup Z^P(s-1) \\ &\rho_i^* = l \qquad l \in Z'. \end{aligned} \tag{2.12}$$

The steady flux of random walkers is given by

$$q_{ij} = (\chi_i \rho_i \alpha_{ij} - \chi_j \rho_j \alpha_{ji}) / \Delta t \tag{2.13}$$

where Δt denotes the interval between two subsequent injections of the random walkers into the system. If

$$\Delta t = \rho_L / K_L \tag{2.14}$$

then

$$q_{ij} = k_{ij}(\rho_i^* - \rho_j^*) \qquad i, j \in Z \cup Z'. \tag{2.15}$$

Let also $\dot{V}_i$ be the 'divergence' of ρ:

$$\dot{V}_i \equiv -\sum_j q_{ij}. \tag{2.16}$$

Then (2.11) reduces to the natural condition for incompressibility of the steady flux:

$$\forall_{i \in Z^F(s-1)} \qquad \dot{V}_i \equiv -\sum_j q_{ij} = 0. \tag{2.17}$$

Equation (2.11) together with its boundary conditions (2.12) allows us to compute ρ_i^*, q_{ij} and lastly $\dot{V}_i$ for any $i \in Z$ and without introducing random walkers at all. As one can also get

$$\dot{V}_L = -K_L/\rho_L \qquad L \in Z' \tag{2.18}$$

we can see that K_L/ρ_L and then, via (2.10), all ρ_i, $i \in Z^F(s-1)$ could be determined too. One could also derive

$$\dot{V}_k = \rho_k K_L/\rho_L \qquad k \in Z^P(s-1) \tag{2.19}$$

so that ρ_k could be computed at all perimeters nodes as well. Moreover, (2.5), (2.8) and (2.19) yield $p_s(k) = \gamma_{kn}\dot{V}_k(\rho_L/K_L)$. As $\Sigma_k p_s(k) = 1$, we finally get

$$p_s(k) = \gamma_{kc_k}\dot{V}_k / \sum_j{}^P \gamma_{jc_j}\dot{V}_j \qquad k, j \in Z^P(s-1) \tag{2.20}$$

where Σ_j^P denotes the sum over all $j \in Z^P(s-1)$. This equation allows us to define the DLA growth process in a new way. Instead of introducing random walkers, we can, for a given $c(s-1)$, determine $Z^A(s-1)$, $Z^P(s-1)$, $Z^F(s-1)$, $\{\rho_i^*\}$, $\{q_{ij}\}$ and $\{\dot{V}_i\}$ simply by making use of appropriate definitions and equations. Then all non-zero conditional probabilities $p(c(s)|c(s-1))$ are given by (2.20).

3. The CT model

In this section we will use the same symbols for the quantities that have their counterparts in the DLA model.

Consider a network (graph) $G(Z, Z', \hat{A}, \{k_{ij}\}, \{\gamma_{in}\}, M)$ of chambers (*nodes*) $i \in Z \cup Z'$ and tubes (bonds) with $k_{ij} = k_{ji} > 0$ being their hydrodynamic conductivities (*weights*). Each chamber i is assumed to be made of $M \geqslant 1$ subchambers† $i^0, \ldots, i^{M-1}$, their volumes $\phi_i^0, \ldots, \phi_i^{M-1}$ being random variables with probability densities $\pi_i^0, \ldots, \pi_i^{M-1}$, so that the probability density π_i of the whole chamber volume ϕ_i may be expressed by

$$\pi_i(\phi_i) = \int_0^\infty \ldots \int_0^\infty \pi_i^0(\phi_i^0) \ldots \pi_i^{M-1}(\phi_i^{M-1})\delta(\phi_i^0 + \ldots + \phi_i^{M-1} - \phi_i)\, \mathrm{d}\phi_i^0 \ldots \mathrm{d}\phi_i^{M-1}. \tag{3.1}$$

† The concept of subchambers arises from the following reasoning. If we simulated the CT model on a computer, in each experiment we could generate any chamber volume ϕ_i as the sum of 'subchamber volumes' ϕ_i^n, chosen in accordance with some simple probability density functions π_i^n, and keep them all in the memory. In this way it could be possible to give direct sense to the statement 'subchamber i has just been completely filled with the driving fluid'. Of course, as the only relevant quantities refer to whole chambers, we could also use a different random-number generator, and the results must remain the same. However, the concept of subchambers proves to be a good tool for dealing with complicated chamber volume probability densities π_i.

We assume that $(\gamma_{in})^{-1}$ are the mean values of the subchamber volumes, i.e. $(\gamma_{in})^{-1} = \langle \phi_i^n \rangle$, $n = 0 \ldots M-1$. If a quantity corresponds to a subchamber and its value may differ from the value referring to the whole chamber, it will be distinguished either by an overbar (e.g. $\bar{\gamma}_i$, $\bar{Z}$), or by writing explicitly the domain of the index (e.g. π_i, $i \in \bar{Z}$). Z' is a special, one-element set (the outer space) to which a fluid can be pushed from the system Z. We also maintain the definition of K_i as well as the notation Σ_j and Σ_j^{P} introduced after equations (2.2) and (2.20), respectively.

The random nature of ϕ_i may be taken into account by making many experiments with $\{\phi_i\}_{i \in Z}$ chosen in each experiment in accordance with the probability distributions π_i, $i \in Z$. In each experiment we may introduce another function $V(t, j) \equiv V_j(t)$ which would correspond to the volume occupied by the *driven* viscous fluid at the time t and at the chamber $j \in Z$. Given $V(t)$, we can divide Z into three disjoint subsets: the inviscid phase (the aggregate) $Z^{\mathrm{A}}(t)$, the viscous phase (the free space) $Z^{\mathrm{F}}(t)$ and the interface (the perimeter) $Z^{\mathrm{P}}(t)$:

$$\begin{aligned} Z^{\mathrm{A}}(t) &\equiv \{i \in Z\colon V_i(t) = 0\} \\ Z^{\mathrm{F}}(t) &\equiv \{i \in Z - Z^{\mathrm{A}}(t)\colon \forall_j A_{ij} = 1 \to j \notin Z^{\mathrm{A}}(t)\} \\ Z^{\mathrm{P}}(t) &\equiv Z - Z^{\mathrm{A}}(t) - Z^{\mathrm{F}}(t). \end{aligned} \tag{3.2}$$

Again given one of these subsets, the other two can be easily obtained.

To describe the rules of flow in the CT model, we introduce another function P (pressure). Its values will be denoted by $P_i(t)$, where $i \in Z \cup Z'$ and $t \in R$. This function fulfils, by definition, the following conditions:

$$i \in Z^{\mathrm{A}}(t) \cup Z^{\mathrm{P}}(t) \to P_i(t) = 1 \tag{3.3a}$$

$$i \in Z' \to P_i(t) = 0 \tag{3.3b}$$

$$i \in Z^{\mathrm{F}}(t) \to P_i(t) = \sum_j P_j(t) k_{ji} / K_i \tag{3.3c}$$

and so it is completely determined by the state of the aggregate $Z^{\mathrm{A}}(t)$, the topology of the network G, and by the hydrodynamic conductivities $\{k_{ij}\}$. Equation (3.3a) states that the driving fluid is inviscid (no pressure drop along tubes filled with this fluid), and that the driving pressure is equal to 1. The next equation states that the pressure outside the system is equal to 0. The last of the above equations says that the driven fluid is Newtonian and incompressible. To see this let us define a volume flux q_{ij}:

$$q_{ij}(t) \equiv k_{ij}(P_i(t) - P_j(t))/\mu \qquad i, j \in Z \cup Z' \tag{3.4}$$

where μ is some positive constant (viscosity of the driven fluid). Equation (3.4) is simply Darcy's law relating the volume flux of a Newtonian fluid to the pressure drop along a tube. Now we can see that (3.3c) is equivalent to

$$\forall_{i \in Z^{\mathrm{F}}} \sum_j q_{ij}(t) = 0 \tag{3.5}$$

i.e. to the condition of incompressibility of the driven fluid. The speed at which one fluid replaces the other, or equivalently $\mathrm{d}V_i(t)/\mathrm{d}t$, is given by

$$\dot{V}_i(t) = \sum_j q_{ij}(t) \qquad i \in Z \cup Z'. \tag{3.6}$$

Thus, in each experiment with fixed $\{\phi_i\}$, if we knew $\{V_i(t)\}$, we would be able to find out $Z^{\mathrm{A}}(t)$ and then $\{\dot{V}_i(t)\}$—this means that $V(t)$ would be in each experiment a deterministic process.

As assumed before, each chamber consists of M subchambers which are, however, purely theoretical objects introduced only to simplify our considerations. To avoid misunderstandings we shall denote the set of all subchambers by $\bar{Z}$ and divide it into $\bar{Z}^{A}(t)$, $\bar{Z}^{P}(t)$ and $\bar{Z}^{F}(t)$ by replacing $V_i(t)$ by $\bar{V}_i(t)$ in (3.2).

As the fluids flow, subsequent subchambers are completely filled up with the driving fluid so that the aggregate $\bar{Z}^{A}(t)$ grows. We will assume that for every chamber i none of its subchambers i^k can be filled unless subchambers $i^1, \ldots, i^{k-1}$ were previously completely filled with the driving fluid. However, as for any t there is always a finite number of interfacial subchambers and the chamber volumes may be any real positive numbers, we may assume that it is impossible for two different subchambers to be fully emptied from the driven fluid at the same time, so that for every flow process there is a sequence of moments $t_0 < t_1 < \ldots < t_s < \ldots$ at which successive subchambers are completely filled with the driving inviscid fluid. At any $t_{s-1} \leqslant t < t_s$ the aggregate $\bar{Z}^{A}(t)$ does not change and so for any $i, j \in Z$ the values of $P_i(t)$, $q_{ij}(t)$ and $\dot{V}_i(t)$ are constant between t_{s-1} and t_s. This property discretizes the problem in a natural way.

Having in mind the similarity of patterns observed in the DLA and in a two-phase flow through porous media (the so-called viscous fingering phenomenon), we may pose the following question: what is the probability that in a sequence of experiments an aggregate $\bar{Z}^{A}(t_{s-1})$ will grow at a subchamber k belonging to the interface $\bar{Z}^{P}(t_{s-1})$? However, the full description of such a probability should depend on the probability densities ν_i^{s-1} of $\bar{V}_i(t_{s-1})$, $i \in \bar{Z}$. To see this let us assume that in an experiment the state of the system at some t_{s-1} is given by $\bar{V}_i(t_{s-1})$, $i \in \bar{Z}$. The subchamber k which will be the first to be attached to $\bar{Z}^{A}$ can be easily determined from the condition $\tau_k = \min\{\tau_i\}$, $i \in \bar{Z}^{P}(t_{s-1})$, where $\tau_i \equiv \bar{V}_i(t_{s-1})/\dot{V}_i(t_{s-1})$ is the time of filling up the volume $\bar{V}_i(t_{s-1})$ at the speed $\dot{V}_i(t_{s-1})$. Therefore, in a sequence of experiments, the probability that a subchamber k will be the next one completely flooded by the driving fluid equals to $\mathrm{prob}(\tau_k = \min\{\tau_i\})$, $i \in \bar{Z}^{P}(t_{s-1})$, or

$$p_s(k) = \int_0^\infty \nu_k^{s-1}(\bar{V}_k)\left(\prod_i{}' \int_{\dot{V}_i \bar{V}_k/\dot{V}_k}^\infty \nu_i^{s-1}(\bar{V}_i)\,\mathrm{d}\bar{V}_i\right)\mathrm{d}\bar{V}_k \tag{3.7}$$

where the product is taken over all $i \in \bar{Z}^{P}(t_{s-1})$ except k. If at t_s the aggregate grows at $k \in \bar{Z}^{P}(t_{s-1})$, then the new probability distributions ν_i^s will be given by

$$\nu_i^s(x) = \pi_i(x) \tag{3.8}$$

for $i \in \bar{Z}^{P}(t_s) - \bar{Z}^{P}(t_{s-1})$. For $i \in \bar{Z}^{P}(t_s) \cap \bar{Z}^{P}(t_{s-1})$ they will read

$$\nu_i^s(x) = B^{-1} \int_0^\infty \nu_i^{s-1}(x + y \cdot \dot{V}_i/\dot{V}_k)\nu_k^{s-1}(y)\,\mathrm{d}y \tag{3.9}$$

where B is the normalizing factor equal simply to $\mathrm{prob}(\tau_k < \tau_i)$,

$$B = \int_0^\infty \int_{y \cdot \dot{V}_i/\dot{V}_k}^\infty \nu_i^{s-1}(x)\nu_k^{s-1}(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.10}$$

Now we can see the main difficulty of the problem: to compute $p_s(k)$ we must have all the functions ν_i^{s-1}, $i \in \bar{Z}^{P}(t_{s-1})$, which may depend on all $\nu_j^{s-2}, \ldots, \nu_j^0 \equiv \pi_j$, $j \in \bar{Z}^{A}(t_{s-1}) \cup \bar{Z}^{P}(t_{s-1})$ in a rather complicated way. Moreover, ν_i^s might depend on the whole 'history' of the aggregate. However, there is a special choice of π_i for which this obstacle vanishes in probably the most natural way.

Assume that all π_i, $i \in \bar{Z}$, are given by the exponential distributions $\xi_a(x)$, where

$$\xi_a(x) \equiv \begin{cases} a \exp(-ax) & x > 0 \\ 0 & x \leqslant 0 \end{cases} \tag{3.11}$$

and $a \equiv \bar{\gamma}_i$ for each $i \in \bar{Z}$. If at some moment $\nu_i^{s-1} \equiv \xi_a$, then (3.9) and (3.10) yield the fundamental 'short-memory' property of the exponential distribution

$$\nu_i^s = \nu_i^{s-1} \tag{3.12}$$

for *any* functional form of ν_k^{s-1}. As for all subchambers initially belonging to $\bar{Z}^F(t_0)$ we have $\nu_i \equiv \pi_i$, iterating (3.12) we get

$$\nu_i^s = \nu_i^{s-1} \ldots = \nu_i^0 \equiv \pi_i \qquad i \in \bar{Z}^F(t_0). \tag{3.13}$$

Now we can conclude that if all π_i are the exponential distributions (3.11), then $p_s(k)$ (see (3.7)) depends only on $\{\pi_i\}$, which are the same in each experiment, and on $\{\dot{V}_i\}$, which implicitly depend only on the state of the aggregate $\bar{Z}^A(t_{s-1})$. Thus, with such choice of $\{\pi_i\}$, $i \in \bar{Z}$, $p_s(k)$ depends only on $\bar{Z}^A(t_{s-1})$ and k. This means that the growth of the inviscid phase in a random medium may be considered as a Markov process. For the exponential distributions of $\{\pi_i\}$, (3.13) and (3.7) yield

$$p_s(k) = \bar{\gamma}_k \dot{V}_k / \sum_j{}^P \bar{\gamma}_j \dot{V}_j \qquad k, j \in \bar{Z}^P(t_{s-1}) \tag{3.14}$$

which is the probability that the aggregate $\bar{Z}^A(t_{s-1})$ will grow at a perimeter subchamber k. In terms of chambers this equation reads

$$p_s(k) = \gamma_{kc_k} \dot{V}_k / \sum_j{}^P \gamma_{jc_j} \dot{V}_j \qquad k, j \in Z^P(t_{s-1}) \tag{3.15}$$

which is the probability that, if the aggregate $\bar{Z}^A(t_{s-1})$ is given, the next subchamber flooded by the driving fluid belongs to the chamber k.

4. Equivalence of the models

Comparing (2.20) and (3.15) we conclude that with the exponential choice of $\{\pi_i\}$, $i \in \bar{Z}$, both the DLA and the CT processes are not only Markovian, but have the same analytical form for the transition probabilities $p_s(k)$. Comparing equations and definitions one immediately arrives at the conclusion that the satisfactory conditions for the equivalence of the two models are:

(i) $\pi_j^n = \xi_a$, where $a \equiv \gamma_{jn}$;

(ii) the values of γ_{jn}, $\dot{V}_j$ and k_{ij} in the CT model should be proportional to their counterparts in the DLA model;

(iii) $(M)_{CT} = (M)_{DLA}$ $\qquad (1 - P_j)_{CT} = (\rho_j^*)_{DLA}$;

(iv) both systems have the same topology (isomorphous sets Z and the contiguity matrices $\hat{A}$).

It is important to notice that in point (ii) we do not require equality, but merely proportionality of the appropriate values. This is the consequence of the fact that multiplication of all k_{ij} or γ_{in} by the same factor would result only in slowing down or speeding up the processes, but could not change their conditional probabilities. thus, though in the DLA model all γ_{jn} should be less than or equal to 1 (they are interpreted as probabilities), no such restriction applies to the CT model. Moreover, as $\dot{V}_j$ and P_j (or $1 - \rho_j^*$) are fully determined by the parameters of the models, all those conditions may be written briefly as

$$\pi_j^n = \xi_a \qquad a \equiv \gamma_{jn} \tag{4.1}$$

$$G_{DLA}(Z, Z', \hat{A}, \{k_{ij}\}, \{\gamma_{jn}\}, M) \simeq G_{CT}(Z, Z', \hat{A}, \{k_{ij}\}, \{\gamma_{jn}\}, M) \tag{4.2}$$

where '$\simeq$' indicates isomorphism. The first of the above conditions, however, refers to the notion of subchambers, which are auxiliary objects. Physically meaningful quantities are chamber volume probability densities, which may be easily derived from (3.1). We will consider here only two simple cases:

(i) $\gamma_{i0} = \ldots = \gamma_{iM-1} \equiv \gamma_i$, then

$$\pi_i(x) = \begin{cases} \gamma_i(\gamma_i x)^{M-1} \exp(-\gamma_i x)/(M-1)! & x > 0 \\ 0 & x \leqslant 0. \end{cases} \tag{4.3}$$

(ii) $\forall_{n \neq m} \gamma_{in} \neq \gamma_{im}$, then

$$\pi_i(x) = \begin{cases} \sum_{n=0}^{M-1} \beta_{in}\gamma_{in} \exp(-\gamma_{in}x) & x > 0 \\ 0 & x \leqslant 0 \end{cases} \tag{4.4}$$

where

$$\beta_{in} \equiv \prod_{m=0, m \neq n}^{M-1} \gamma_{im}/(\gamma_{im} - \gamma_{in}) \tag{4.5}$$

are some constants.

Thus the class of chamber volume probability densities π_i for which there exists an equivalent DLA-like model is quite large and this suggests that these two processes belong to the same universality class (i.e. their fractal properties are the same) for any choice of π_i, $i \in Z$.

We will finish this section with a few straightforward applications of the two models equivalence.

4.1. The limit $M \to \infty$

Assuming in (4.3) $\gamma_i = M$, we obtain

$$\lim_{M \to \infty} \pi_i(x) = \delta(x-1).$$

This means that in this limit the chamber volumes are no longer random numbers—they all are equal to 1. If also the system is uniform ($k_{ij} = k = \text{const}$), and the graph G corresponds to a regular lattice (e.g. square or hexagonal), then the CT model is a discrete-lattice approximation of a two-phase flow in a uniform deterministic medium (e.g. in a Hele-Shaw cell). But as the CT model is equivalent to the DLA algorithm, we conclude that also the 'mean-field' limit ($M \to \infty$) of the classical DLA ($k_{ij} = \text{const}$, $\gamma_{in} = \text{const}$, regular lattice) is equivalent to the Saffman–Taylor problem of a two-phase flow (an inviscid fluid driving a viscous one) in a Hele-Shaw cell.

4.2. Noise and entropy

It is well known (Tang 1985, Nittmann and Stanley 1986) that the parameter M controls the noise level in the DLA algorithm. The noise is the biggest for $M = 1$, and vanishes monotonically as M goes to infinity. On the other hand, however, Chan *et al* (1988) showed that this is just the exponential chamber volume probability density, for which a random medium with a fixed mean pore size has the biggest entropy. Looking at (4.3) or (4.4) we immediately conclude that in the CT model such a chamber volume probability density can occur only for $M = 1$. As M goes to infinity, the randomness

(entropy) of a porous medium vanishes monotonically, and so we can conclude that there is a natural correspondence between the noise in the DLA model and the randomness (entropy) of a porous medium in the CT model.

4.3. DLA as a real-time process

It would be interesting to know if the DLA algorithm could be employed to investigate time-dependent aspects of the viscous fingering phenomena. This problem has been explored by Måløy *et al* (1987), who compared experimental results with computer simulations of the DLA in the zero-concentration limit, finding good correspondence of the results. They simulated the flow of time by incrementing the real-time variable by $\Delta t' = 1/N_L$ each time a new particle was injected into the system, N_L being the number of the system nodes to which those random walkers could be launched from Z' (boundary nodes). In fact this is equivalent to (2.14), which reads $\Delta t = \rho_L / K_L$. To see this we must notice two facts. Firstly, in our approach Δt denotes the mean interval in which subsequent particles join the cluster, while Måløy *et al* by $\Delta t'$ meant the mean time between subsequent entering Z by random particles. However, as ρ_L is the mean number of times each particle visits Z', $\Delta t = \rho_L \Delta t'$. The other fact we should realize is that if all the boundary nodes of Z may be reached from Z' with the same probabilities (which was assumed by Måløy *et al*), then $k_{iL} = \text{const}(i)$ and so $K_L \propto N_L$.

Now assume that all $\gamma_{in} = 1$, what means that the random walker must be incorporated into the aggregate each time it arrives at the perimeter $Z^P(s-1)$ or, on the other hand, that the mean volume of each subchamber is the same and equal to 1. Let also M and $\{k_{ij}\}$ be the same in the two models. Then in the DLA algorithm it takes $\Delta t = \rho_L / K_L$ to attach a single particle into the cluster. In the CT model the driving fluid is being pushed into the system at the speed $\dot{V}_L(t_{s-1})$, and on average it should take $(\Delta t)_{CT} = 1/\dot{V}_L(t_{s-1})$ to fill up a single subchamber, which, due to the short-memory property of the exponential distribution, should always have the mean unfilled volume equaling to $1/\gamma_{in} = 1$, even if it has been partially filled with the driving fluid. However, as the same equations with the same boundary conditions should always give the same results, $(1-\rho_i^*)_{DLA} = (P_i)_{CT}$, $(-q_{ij})_{DLA} = (q_{ij}/\mu)_{CT}$ and $(\dot{V}_L)_{DLA} = (\dot{V}_L/\mu)_{CT}$, so that using (2.18) we finally write

$$(\Delta t)_{CT} = 1/(\dot{V}_L)_{CT} = \mu(\rho_L/K_L)_{DLA} = \mu(\Delta t)_{DLA}.$$

Thus the DLA may be used, via the CT model, to investigate time-dependent aspects of the viscous fingering phenomena, at least in the most common case $\gamma_{in} = \text{const}$, simply by incrementing the time variable by μ/K_L each time a random walker enters the system Z.

5. Conclusions

We have proved that the extended DLA model is mathematically equivalent to the CT model with properly chosen parameters. Our proof is valid for any lattice (graph) with any space dimensionality. Employing some concepts presented by the authors of the CT model, we completed the proof that the hydrodynamic process described by their model is Markovian. Moreover, we were able to generalize our considerations to systems with any values of M, $\{k_{ij}\}$ and $\{\gamma_{ij}\}$.

The first of these parameters, M, controls the level of the noise in the DLA algorithm and, as we showed in section 4, may be related to the randomness of a porous medium in the CT model. In contrast to Chan *et al* (1988), we let $\{k_{ij}\}$ take any value because we believe that in the two-phase flow problem the distribution of the local hydrodynamic conductivities is at least as important as the probability distribution of the chamber volumes (local porosities)—compare results obtained by Meakin (1987), and by Nittmann and Stanley (1986). Lastly we have introduced $\{\gamma_{in}\}$ to enlarge the family of the chamber volume probability densities π_i, for which an equivalent DLA-like model can be constructed, and to give a firmer basis to the hypothesis that the DLA and the CT models belong to the same universality class.

We managed to consider the case $M > 1$ by introducing auxiliary objects called 'subchambers' and utilizing the 'short-memory' property of the exponential probability distribution. Taking the limit $M \to \infty$ we showed the equivalence of the mean-field limit of the DLA and the Saffman–Taylor problem of two-phase flow in a Hele-Shaw cell.

We also showed that the DLA model may be employed to investigate time-dependent aspects of the CT model, at least in the case $\gamma_{in} = \text{const}$. The simplest way to achieve this is to increment the time variable by μ/K_L, $l \in Z'$, each time the random walker is entering Z.

In the future the equivalence of the two models may be employed to join closer those apparently different areas of physics—hydrodynamics in porous media and stochastic growth processes. One could for instance try to find out the exact relation between the DLA 'fractal' exponents and some features of the flow through a porous medium. Moreover, the CT model is rather simple and could be easily modified so as to take account of gravitational or capillary effects.

Our work reveals also the fundamental significance of the probability density of the pore volumes, and it seems that this aspect of flow through porous media requires a closer experimental exploration.

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